Frequency-Dependent Relaxation Time of Electron-Phonon Systems in a Homogeneous Magnetic Field

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The problem of electromagnetic-wave absorption in a simple model of metal, consisting of electrons and phonons in the presence of a uniform static magnetic field, is investigated by a quantum-kinetic description. The treatment, which stems from a proper time-dependent transport equation, does not have the time-scale restriction of the usual transition-probability approach and does give a proper description of the time-dependent collective effects in a self-consistent way.

I. INTRODUCTION

IN this paper we are concerned with the problem of calculating the frequency-dependent conductivity and relaxation time of a simplified model of a metal in the presence of a homogeneous, static magnetic field. In this model, our system is composed of electrons and phonons, placed in a homogeneous static magnetic field and a homogeneous electric field given by $Ee^{-i\omega t}$. Our solution is valid in the high-frequency limit, which means for the conductivity parallel to the direction of magnetic field, ω must be large compared with ν; for the conductivity perpendicular to the direction of the magnetic field, $|\omega-\omega_c|\gg\nu$. Here ν is the collision frequency which is of the order of the phonon frequency and $\omega_c=eB/mc$ is the electron cyclotron frequency. We also restrict ourselves to magnetic fields such that $\hbar\omega_c\ll E_F$, where E_F is the Fermi energy of the electron gas.

Our treatment, based on the kinetic description, takes into account the finite duration of encounters in a self-consistent fashion which properly include collective effects.

Our solution is also based on the assumption that the phonon distribution is not affected by the electron-phonon interaction in the presence of the electric field. This approximation is valid for applied frequencies $\omega\gg\omega_k$, the phonon frequency.

We obtain simple expressions for the conductivity which reduce to those of Ron[1] in the absence of the magnetic field. It is expected that our results could find applications in the study of absorption of the electromagnetic waves in metals.

II. THE FORMULATION

Following Bardeen and Pines,[2] we assume a monoatomic crystal of n ions and n valence electrons per unit volume. We introduce phonon coordinates to represent the ion motion, and second quantization representation for the electrons. The phonons are, in the long-wavelength limit, either longitudinal or transverse. We further assume that only the longitudinal phonons interact with the electrons. This approximation amounts to neglect of anisotropic effects which couple the electron density fluctuation with transverse phonons. Such effects greatly complicate the equations and are probably important only for detailed quantitative calculations.

The longitudinal phonons are described by the Hamiltonian

$$H_{\mathrm{ph}}=\tfrac{1}{2}\sum_{\mathbf{k}(\mathrm{zone})} P_{\mathbf{k}}{}^{\dagger}P_{\mathbf{k}}+\omega_{\mathbf{k}}{}^{2}Q_{\mathbf{k}}{}^{\dagger}Q_{\mathbf{k}},$$

where $Q_{\mathbf{k}}$ and $P_{\mathbf{k}}$ are the normal coordinates and the conjugate momenta respectively for phonons in the k state obeying the usual commutation relations. Here $\omega_{k}{}^{2}$ represents the renormalized phonon spectrum (thereafter simply called the phonon spectrum) which is determined not only by the ion-ion interaction but also by the ion-electron interaction. [See, for example, Eq. (27) or Ref. 1.]

We shall describe the electron dynamics by an equation of motion for the Wigner distribution function. One has to, therefore, decouple the electrons from the phonons in the density matrix for the electron-phonon system. This is achieved by adopting the adiabatic approximation, assuming that the electrons move as though the ion lattice were fixed in their instantaneous positions. One can then have a density matrix just for the electrons which contains the phonon coordinates as parameters. This approximation is expected to be valid when $\omega\gg\omega_k$ so that the phonons cannot quite follow the high-frequency electron-density fluctuations. We next put our electron-phonon system under the influence of a prevailing spatially uniform electric field $\mathbf{E}$ oscillating in time at the frequency ω, and a homogeneous static magnetic field $\mathbf{B}$. We are only interested in the high-frequency region as we defined in the Introduction. This implies that we may systematically neglect the electron-electron correlations including their exchange effects.[3]

We also limit ourselves to weak magnetic fields, i.e., $\hbar\omega_c\ll E_F$, where ω_c is the electron cyclotron frequency and E_F is the Fermi energy. This is not a severe restriction on the field in the case of metals. In this limit one may neglect the quantization of the circular orbits of

[1] A. Ron, Phys. Rev. **131**, 2041 (1963). The readers should refer to this paper for a survey of previous works.

[2] J. Bardeen and D. Pines, Phys. Rev. **99**, 1140 (1955).

[3] C. Oberman, A. Ron, and J. Dawson, Phys. Fluids **5**, 1514 (1963).

the electrons.[4] Thus, we obtain the equation of motion for the Wigner distribution function $F(\mathbf{R},\mathbf{p},t)$[4] in the laboratory frame:

$$\frac{\partial F}{\partial t}+\frac{\mathbf{p}}{m}\frac{\partial F}{\partial \mathbf{R}}-e\left[\mathbf{E}e^{-i\omega t}+\frac{1}{mc}\mathbf{p}\times\mathbf{B}\right]\frac{\partial F}{\partial \mathbf{p}}$$

$$=ie\int d\mathbf{r}E^{-i\mathbf{p}\cdot\mathbf{r}}[\Phi(\mathbf{R}+\tfrac{1}{2}\mathbf{r})-\Phi(\mathbf{R}-\tfrac{1}{2}\mathbf{r})]$$

$$\times\int\frac{d\mathbf{p}'}{(2\pi)^3}e^{i\mathbf{p}'\cdot\mathbf{r}}F(\mathbf{R},\mathbf{p}',t),\quad (1)$$

where

$$F(\mathbf{R},\mathbf{p},t)=\frac{1}{(2\pi)^3}\int d\mathbf{r}e^{i\mathbf{p}\cdot\mathbf{r}}\,\mathrm{Tr}$$

$$\times\left\{\rho\psi^\dagger\left(\mathbf{R}-\frac{\mathbf{r}}{2},t\right)\psi\left(\mathbf{R}+\frac{\mathbf{r}}{2},t\right)\right\}.\quad (2)$$

Here $\psi^\dagger(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$ are, respectively, the creation and annihilation operators for the electrons, and ρ, the many-electron density matrix in the Heisenberg representation. Also, $-e$ and m denote the charge and mass of the electron, c is the speed of light and $\hbar$, the Planck constant, is taken to be unity. In Eq. (1), $\Phi(\mathbf{R},t)$ is given by

$$\Phi(\mathbf{R},t)=-e\int d\mathbf{R}'d\mathbf{p}\,|\,\mathbf{R}'-\mathbf{R}\,|^{-1}F(\mathbf{R}',\mathbf{p},t)+U_{\mathrm{ion}}(\mathbf{R},t)\quad (3)$$

$$U_{\mathrm{ion}}(\mathbf{R},t)=\sum_i \boldsymbol{\nabla}v(\mathbf{R}-\mathbf{r}_i{}^0)\cdot\delta\mathbf{r}_i(t),\quad (4)$$

where the first term in Eq. (3) is the self-consistent field of the electrons and the second term arises from the interaction of the electrons with the vibrating lattice. Here $\mathbf{r}_i{}^0$ denotes the equilibrium position of the ion, and $\delta\mathbf{r}_i(t)$ denotes the "renormalized" deviation from the equilibrium. This means that the influence of the electron gas on the motion of the ions is already taken into account in $\delta\mathbf{r}_i(t)$. In Eq. (1), any interaction due to the spin of the particles is systematically neglected.

III. THE APPROXIMATION METHOD

To facilitate solving the coupled equations (1) and (3), we perform the transformation to an oscillating frame in which the electrons do not see the electric field explicitly[5]:

$$\boldsymbol{\varrho}=\mathbf{R}+\boldsymbol{\zeta}e^{-i\omega t},$$

$$\mathbf{q}=\mathbf{p}-i\omega m\boldsymbol{\zeta}e^{-i\omega t},\quad (5)$$

$$t=t,$$

[4] Yu. L. Klimontovich and V. P. Silin, Usp. Fiz. Nauk 70, 247 (1960) [English transl.: Soviet Phys.—Usp. 3, 84 (1960)].

[5] C. Oberman and A. Ron, Phys. Rev. 130, 1291 (1963).

with

$$\boldsymbol{\zeta}=-\boldsymbol{\varepsilon}+\left(1-\frac{\omega_c{}^2}{\omega^2}\right)^{-1}\left\{i\frac{\omega_c}{\omega}(\boldsymbol{\varepsilon}\times\mathbf{b})-\frac{\omega_c{}^2}{\omega^2}[\mathbf{b}\times(\boldsymbol{\varepsilon}\times\mathbf{b})]\right\},$$

$$\boldsymbol{\varepsilon}=e\mathbf{E}/m\omega^2,$$

$$\omega_c=e|\mathbf{B}|/mc,\quad \mathbf{b}=\mathbf{B}/|\mathbf{B}|.$$

Equations (1) and (3) become

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{q}}{m}\frac{\partial}{\partial\boldsymbol{\varrho}}-\omega_c(\mathbf{q}\times\mathbf{b})\cdot\frac{\partial}{\partial\mathbf{q}}\right]\tilde{F}(\boldsymbol{\varrho},\mathbf{q},t)$$

$$=ie\int d\mathbf{r}e^{-i\mathbf{q}\cdot\mathbf{r}}[\tilde{\Phi}(\boldsymbol{\varrho}+\tfrac{1}{2}\mathbf{r},t)-\tilde{\Phi}(\boldsymbol{\varrho}-\tfrac{1}{2}\mathbf{r},t)]\frac{1}{(2\pi)^3}$$

$$\times\int d\mathbf{q}'e^{i\mathbf{q}'\cdot\mathbf{r}}\tilde{F}(\boldsymbol{\varrho},\mathbf{q}',t),\quad (6)$$

$$\tilde{\Phi}(\boldsymbol{\varrho},t)=U_{\mathrm{ion}}(\boldsymbol{\varrho}-\boldsymbol{\zeta}e^{-i\omega t},t)$$

$$-e\int d\boldsymbol{\varrho}'d\mathbf{q}\,|\,\boldsymbol{\varrho}-\boldsymbol{\varrho}'\,|^{-1}\tilde{F}(\boldsymbol{\varrho}',\mathbf{q},t)\quad (7)$$

with

$$\tilde{F}(\boldsymbol{\varrho},\mathbf{q},t)=F(\boldsymbol{\varrho}-\boldsymbol{\zeta}e^{-i\omega t},\mathbf{q}+i\omega m\boldsymbol{\zeta}e^{-i\omega t},t).$$

We now assume that the right-hand side of Eq. (6) causes only a small perturbation on the equilibrium solution of that equation,

$$f_0(\mathbf{p})\equiv\tilde{F}_0(\mathbf{p})=\frac{2}{(2\pi)^3n_0}\left\{\exp\left[\beta\left(\frac{p^2}{2m}-\mu\right)\right]+1\right\}^{-1},\quad (8)$$

where n_0 is the average electron density, μ is the chemical potential of the noninteracting electrons, $\beta=1/kT$ and $f(\mathbf{p})$ is normalized to one upon integration over $\mathbf{p}$. (The factor 2 comes from the summation over the spin components.) This implies that the vibration of the lattice causes only a small effect. The equations for $\tilde{f}$ and $\tilde{\Psi}$, the small departures from equilibrium, are

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{q}}{m}\cdot\frac{\partial}{\partial\boldsymbol{\varrho}}-\omega_c(\mathbf{q}\times\mathbf{b})\cdot\frac{\partial}{\partial\mathbf{q}}\right]\tilde{f}(\boldsymbol{\varrho},\mathbf{q},t)$$

$$=ie\int\frac{d\mathbf{q}'}{(2\pi)^3}f_0(q')\int d\mathbf{r}e^{-i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{r}}$$

$$\times[\tilde{\Psi}(\boldsymbol{\varrho}+\tfrac{1}{2}\mathbf{r},t)-\tilde{\Psi}(\boldsymbol{\varrho}-\tfrac{1}{2}\mathbf{r},t)]\quad (9)$$

and

$$\tilde{\Psi}(\boldsymbol{\varrho},t)=U_{\mathrm{ion}}(\boldsymbol{\varrho}-\boldsymbol{\zeta}e^{-i\omega t},t)$$

$$-en_0\int d\boldsymbol{\varrho}'d\mathbf{q}\,|\,\boldsymbol{\varrho}-\boldsymbol{\varrho}'\,|^{-1}\tilde{f}(\boldsymbol{\varrho}',\mathbf{q},t).\quad (10)$$

If we denote the Fourier transform of a function

$f(\varrho)$ by

$$f(\mathbf{k}) = \frac{1}{(2\pi)^3} \int d\varrho e^{i\mathbf{k}\cdot\rho} f(\varrho)$$

and Fourier-analyze Eqs. (9) and (10), we have

$$\left[\frac{\partial}{\partial t} - \frac{i\mathbf{k}\cdot\mathbf{q}}{m} - \omega_c(\mathbf{q}\times\mathbf{b})\cdot\frac{\partial}{\partial \mathbf{q}}\right]\tilde{f}(\mathbf{k},\mathbf{q},t)$$

$$= ie\tilde{\Psi}(\mathbf{k},t)\left[f_0\left(\mathbf{q}+\frac{\mathbf{k}}{2}\right)-f_0\left(\mathbf{q}-\frac{\mathbf{k}}{2}\right)\right]\quad(11)$$

and

$$\tilde{\Psi}(\mathbf{k},t) = -\frac{4\pi e n_0}{k^2}\int d\mathbf{q}\tilde{f}(\mathbf{k},\mathbf{q},t)$$

$$+\exp[i\mathbf{k}\cdot\zeta e^{-i\omega t}]U_{\text{ion}}(\mathbf{k},t),\quad(12)$$

where

$$U_{\text{ion}}(\mathbf{k},t) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{R}}U_{\text{ion}}(\mathbf{R},t)d\mathbf{R}.$$

We now introduce the phonon coordinate $Q_\mathbf{k}(t)$ into $U_{\text{ion}}(\mathbf{k},t)$ by the following equation.

$$U_{\text{ion}}(\mathbf{k},t) = -\frac{1}{e}Q_\mathbf{k}(t)v_\mathbf{k},\quad(13)$$

where

$$v_\mathbf{k} = -(nM)^{-1/2}\int d\mathbf{R}\Phi_{p+k}{}^*(\mathbf{R})$$

$$\times\left[\sum_i \varepsilon_\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{R}}v(\mathbf{R}-\mathbf{r}_i{}^0)e^{i\mathbf{k}\cdot\mathbf{r}_i{}^0}\right]\Phi_p(\mathbf{R})$$

and Φ_p is the $\mathbf{p}$ state.

Since the conductivity is defined by the limit $\varepsilon \to 0$ we shall expand the second term on the right-hand side of Eq. (12) and obtain

$$\tilde{\Psi}(\mathbf{k}t) = \frac{-4\pi e n_0}{k^2}\int d\mathbf{q}f(\mathbf{k},\mathbf{q},t)$$

$$+(1+i\mathbf{k}\cdot\zeta e^{-i\omega t})U_{\text{ion}}(\mathbf{k},t).\quad(14)$$

We proceed by choosing a cylindrical coordinate system, with polar axis along $\mathbf{B}$, in terms of which the rectangular components of $\mathbf{k}$, $\mathbf{q}$, and ε are

$$\mathbf{k} = (k_\perp\cos\alpha,\ k_\perp\sin\alpha,\ k_{||}),$$

$$\mathbf{q} = (q_\perp\cos\Phi,\ q_\perp\sin\Phi,\ q_{||}),$$

$$\varepsilon = (\epsilon_\perp\cos\theta,\ \epsilon_\perp\sin\theta,\ \epsilon_{||}),$$

where the angles are measured from some direction, the x axis, in a plane perpendicular to $\mathbf{B}$. We now introduce the transformation[6]

$$F(\mathbf{k},\mathbf{q}) = e^{-ik_\perp a\ \sin(\Phi-\alpha)}$$

$$\times\sum_n e^{in(\Phi-\alpha)}J_n(k_\perp a)F^{(n)}(\mathbf{k},\mathbf{q}),\quad(15)$$

with the inversion

$$F^{(n)}(\mathbf{k},\mathbf{q}) = \frac{1}{2\pi J_n(k_\perp a)}$$

$$\times\int_0^{2\pi}d\Phi e^{ik_\perp a\ \sin(\Phi-\alpha)}e^{-in(\Phi-\alpha)}F(\mathbf{k},\mathbf{q}),\quad(16)$$

where $a = -q_\perp c/eB$, and the J_n are Bessel functions of the first kind. If we apply this transformation to Eqs. (11) and (14) we obtain

$$\left(\frac{\partial}{\partial t} - \frac{ik_{||}q_{||}}{m} + in\omega_c\right)\tilde{f}^{(n)}(\mathbf{k},\mathbf{q},t)$$

$$= ie\left[f_0{}^{(n)}\left(\mathbf{q}+\frac{\mathbf{k}}{2}\right) - f_0{}^{(n)}\left(\mathbf{q}-\frac{\mathbf{k}}{2}\right)\right]\tilde{\Psi}(\mathbf{k},t),\quad(17)$$

where

$$\tilde{\Psi}(k,t) = \frac{4\pi e n_0}{k^2}\int d\mathbf{q}\sum_n J_n{}^2(k_\perp a)\tilde{f}^{(n)}(\mathbf{k},\mathbf{q},t)$$

$$-\frac{v_\mathbf{k}Q_\mathbf{k}(t)}{e}(1+i\mathbf{k}\cdot\zeta e^{-i\omega t}).\quad(18)$$

It is now convenient to introduce the second quantization representation for the phonons by the equation

$$Q_\mathbf{k}(t) = \frac{1}{(2\omega_k)^{1/2}}[b_\mathbf{k}e^{-i\omega_k t} + b_{-\mathbf{k}}{}^\dagger e^{i\omega_k t}],\quad(19)$$

where ω_k represents the phonon spectrum and $b_\mathbf{k}$, $b_\mathbf{k}{}^\dagger$ are the phonon annihilation and creation operators obeying the usual commutation relations.

We next substitute Eq. (19) into the Eq. (18) for $\tilde{\Psi}(\mathbf{k},t)$. The solution for the coupled Eq. (17) and (18) yields

$$\tilde{n}(\mathbf{k},t) = \int \tilde{f}(\mathbf{k},\mathbf{q},t)d\mathbf{q} = \frac{v_k}{(2\omega_k)^{1/2}}\left\{b_\mathbf{k}\left[\frac{B(\mathbf{k},\omega_k)}{D(\mathbf{k},\omega_k)}e^{-i\omega_k t} + i\mathbf{k}\cdot\zeta\frac{B(\mathbf{k},\ \omega+\omega_k)}{D(\mathbf{k},\ \omega+\omega_k)}e^{-i(\omega+\omega_k)t}\right]\right.$$

$$\left.+b_{-\mathbf{k}}{}^\dagger\left[\frac{B(\mathbf{k},\ -\omega_k)}{D(\mathbf{k},\ -\omega_k)}e^{i\omega_k t} + i\mathbf{k}\cdot\zeta\frac{B(\mathbf{k},\ \omega-\omega_k)}{D(\mathbf{k},\ \omega-\omega_k)}e^{-i(\omega-\omega_k)t}\right]\right\},\quad(20)$$

[6] N. Rostoker, Phys. Fluids **3**, 922 (1960); C. Oberman and F. Shure, *ibid.* **6**, 834 (1963); C. Oberman and A. Ron, Phys. Rev. **130**, 1291 (1963).

where

$$D(\mathbf{k},\omega) = 1 - \frac{4\pi e^2 n_0}{k^2} B(\mathbf{k},\omega),$$ (21)

and

$$B(\mathbf{k},\omega) = \sum_n J_n{}^2(k_\perp a)\sum_{\mathbf{p}} \frac{f_0{}^{(n)}(\mathbf{p}+\mathbf{k}/2) - f_0{}^{(n)}(\mathbf{p}-\mathbf{k}/2)}{(p_{||}k_{||}/m) - n\omega_c + \omega + i\nu}.$$ (22)

Recalling that $f(\mathbf{R},\mathbf{p},t) = \tilde{f}(\varrho,\mathbf{q},t)$, one can easily show that

$$\tilde{n}(\mathbf{k},t) = \exp[i\mathbf{k}\cdot\boldsymbol{\zeta} e^{-i\omega t}]\frac{1}{(2\pi)^3}\int d\mathbf{R} e^{i\mathbf{k}\cdot\mathbf{R}}\int d\mathbf{p} f(\mathbf{R},\mathbf{p},t)$$

so that (23)

$$n(\mathbf{k},t) \simeq (1 - i\mathbf{k}\cdot\boldsymbol{\zeta} e^{-i\omega t})\tilde{n}(\mathbf{k},t).$$

Substituting (20) into (23), one obtains, for the density fluctuation in the laboratory frame,

$$n(\mathbf{k},t) = -\frac{iv_k\mathbf{k}\cdot\boldsymbol{\zeta}}{\Phi_k(2\omega_k)^{1/2}}\left\{ b_k\left[\frac{1}{D(\mathbf{k},\omega_k)} - \frac{1}{D(\mathbf{k},\omega+\omega_k)}\right]e^{-i(\omega+\omega_k)t} + b_{-k}{}^\dagger\left[\frac{1}{D(\mathbf{k},-\omega_k)} - \frac{1}{D(\mathbf{k},\omega-\omega_k)}\right]e^{-i(i-\omega_k)t}\right\}$$

$$+\frac{v_k}{(2\omega_k)^{1/2}}\left\{ b_k\frac{B(\mathbf{k},\omega_k)}{D(\mathbf{k},\omega_k)}e^{-i\omega_k t} + b_{-k}{}^\dagger\frac{B(\mathbf{k},-\omega_k)}{D(\mathbf{k},-\omega_k)}e^{i\omega_k t}\right\}, \quad (24)$$

where

$$\Phi(k) = 4\pi e^2 n_0/k^2.$$

To relate $n(\mathbf{k},t)$ to the current we next Fourier-analyze Eq. (1) and express the electron-lattice interaction in terms of the phonon coordinates, and thus obtain

$$\left[i\frac{\partial}{\partial t} + \frac{\mathbf{k}\cdot\mathbf{p}}{m} - i\omega_c\mathbf{p}\times\mathbf{b}\cdot\frac{\partial}{\partial\mathbf{p}}\right]F\left(\mathbf{p}+\frac{\mathbf{k}}{2}, \mathbf{p}-\frac{\mathbf{k}}{2}, t\right) = +ie\mathbf{E}\cdot\frac{\partial F}{\partial\mathbf{p}}\left(\mathbf{p}+\frac{\mathbf{k}}{2}, \mathbf{p}-\frac{\mathbf{k}}{2}, t\right)e^{-i\omega t}$$

$$-\sum_{\mathbf{k}'}\left[\frac{4\pi e^2}{k'^2}n(\mathbf{k}',t) + v_{\mathbf{k}'}Q_{\mathbf{k}'}(t)\right]\left\{ F\left(\mathbf{p}+\frac{\mathbf{k}}{2}-\mathbf{k}', \mathbf{p}-\frac{\mathbf{k}}{2}, t\right) - F\left(\mathbf{p}+\frac{\mathbf{k}}{2}, \mathbf{p}-\frac{\mathbf{k}}{2}+\mathbf{k}', t\right)\right\}, \quad (25)$$

where $F(\mathbf{p}+\mathbf{k}/2, \mathbf{p}-\mathbf{k}/2, t) = \langle a_{\mathbf{p}-\mathbf{k}/2}{}^\dagger(t)a_{\mathbf{p}+\mathbf{k}/2}(t)\rangle$ is the Fourier transform of $F(\mathbf{R},\mathbf{p},t)$ [see Eq. (2)].

Using the definition of the current density in the second quantization representation, we obtain for the average, homogeneous current density

$$\mathbf{J}(t) = -\frac{e}{m}\sum_{\mathbf{p}} \mathbf{p}F(\mathbf{p},\mathbf{p},t),$$ (26)

where $F(\mathbf{p},\mathbf{p},t) = \lim_{k\to 0} F(\mathbf{p}+\mathbf{k}/2, \mathbf{p}-\mathbf{k}/2, t)$.

By combining Eqs. (25) and (26), one finds

$$i\frac{\partial\mathbf{J}}{\partial t} - i\omega_c\mathbf{b}\times\mathbf{J} = \frac{ie^2 n_0}{m}\mathbf{E}e^{-i\omega t}$$

$$+\frac{e}{m}\sum_{\mathbf{k}'}\mathbf{k}'v_{\mathbf{k}'}Q_{\mathbf{k}'}(t)n(-\mathbf{k}',t) \quad (27)$$

where n_0 is the average electron density and

$$n(\mathbf{k},t) = \sum_{\mathbf{p}} F\left(\mathbf{p}+\frac{\mathbf{k}}{2}, \mathbf{p}-\frac{\mathbf{k}}{2}, t\right).$$

However, limiting ourselves to the lowest order in the plasma parameter (in the high-density approximation) we approximate $n(\mathbf{k},t)$ by Eq. (24). Using Eqs. (27) and (24), we obtain for $\mathbf{J}(\omega)$,

$$\mathbf{J}(\omega) - \frac{i\omega_c}{\omega}\mathbf{b}\times\mathbf{J}(\omega) = \frac{ie^2 n_0}{\omega m}\mathbf{E} + \frac{ie}{m\omega}\sum_{\mathbf{k}}\frac{\mathbf{k}\mathbf{k}\cdot\boldsymbol{\zeta}|v_k|^2}{2\omega_k\Phi(k)}$$

$$\times (b_k{}^\dagger b_k + \tfrac{1}{2})\left\{\left[\frac{1}{D(\mathbf{k},\omega_k)} + \frac{1}{D(\mathbf{k},-\omega_k)}\right]\right.$$

$$\left. -\left[\frac{1}{D(\mathbf{k},\omega+\omega_k)} + \frac{1}{D(\mathbf{k},\omega-\omega_k)}\right]\right\}, \quad (28)$$

where vacuum current has been eliminated.

Following Bloch we shall assume that the phonon system may be treated as if it were in thermal equilibrium. Therefore, we obtain

$$\langle b_k{}^\dagger b_k\rangle + \tfrac{1}{2} = \tfrac{1}{2}\coth(\beta\omega_k/2).$$ (29)

Thus, our final expression for the current is given by

$$\mathbf{J}(\omega) - \frac{i\omega_c}{\omega}\mathbf{b}\times\mathbf{J}(\omega) = \frac{ie^2 n_0\mathbf{E}}{m} - \frac{ie}{m\omega}\sum_\mathbf{k}\frac{\mathbf{k}\mathbf{k}\cdot\boldsymbol{\zeta}|v_\mathbf{k}|^2}{4\omega_k\Phi(k)}$$

$$\times\coth\frac{\beta\omega_k}{2}\left\{\left[\frac{1}{D(\mathbf{k},\omega_k)}+\frac{1}{D(\mathbf{k},-\omega_k)}\right]\right.$$

$$\left.-\left[\frac{1}{D(\mathbf{k},\omega+\omega_k)}+\frac{1}{D(\mathbf{k},\omega-\omega_k)}\right]\right\}. \quad (30)$$

We now utilize right- and left-polarized components of $\mathbf{E}\perp\mathbf{B}$ and find with $J_\pm=J_x\pm iJ_y$ and $J_{||}=J_z$, $E_\pm=E_x\pm iE_y$, and $E_{||}=E_z$, that

$$J_\pm(\omega)=\sigma_\pm(\omega)E_\pm,$$
$$J_{||}(\omega)=\sigma_{||}(\omega)E_{||},$$

where

$$\sigma_\pm(\omega)=\frac{ie^2 n_0}{m\omega}\left(1+\frac{\omega_c}{\omega}\right)^{-1}\left[1-\frac{e^2}{m\omega^2}\left(1\pm\frac{\omega_c}{\omega}\right)^{-1}I_2(\omega)\right], \quad (31)$$

$$\sigma_{||}(\omega)=\frac{ie^2 n_0}{m\omega}\left[1-\frac{e^2}{m\omega^2}I_1(\omega)\right]. \quad (32)$$

The functions I_1 and I_2 are defined by

$$I_1(\omega)=\sum_\mathbf{k}\frac{k_{||}^2|v_k|^2}{e^2 4\Phi(k)\omega_k n_0}$$

$$\times\coth\frac{\beta\omega_k}{2}\left\{\left[\frac{1}{D(\mathbf{k},\omega_k)}+\frac{1}{D(\mathbf{k},-\omega_k)}\right]\right.$$

$$\left.-\left[\frac{1}{D(\mathbf{k},\omega+\omega_k)}+\frac{1}{D(\mathbf{k},\omega-\omega_k)}\right]\right\}, \quad (33)$$

$$I_2(\omega)=\sum_\mathbf{k}\frac{k_\perp^2|v_k|^2}{e^2 8\Phi(k)\omega_k n_0}$$

$$\times\coth\frac{\beta\omega_k}{2}\left\{\left[\frac{1}{D(\mathbf{k},\omega_k)}+\frac{1}{D(\mathbf{k},-\omega_k)}\right]\right.$$

$$\left.-\left[\frac{1}{D(\mathbf{k},\omega+\omega_k)}+\frac{1}{D(\mathbf{k},\omega-\omega_k)}\right]\right\}. \quad (34)$$

Equations (31) and (32) constitute our result for the conductivity tensor. It is expressed in terms of $D(\mathbf{k}\omega)$, the electronic dielectric function in a homogeneous magnetic field within the random-phase approximation. When the magnetic field is weak, i.e., $\omega_c\ll\omega$, the dielectric function $D(\mathbf{k}\omega)$ is independent of the field to the first order in ω_c. In this case, the sole dependence of the conductivity on magnetic field comes from the factors $(1\pm\omega_c/\omega)^{-1}$. This is an uninteresting limit since it could be obtained by simple perturbation methods. Hence, the

more interesting case is in the region of higher magnetic fields, $\omega_c\sim\omega$, but which still satisfy our basic assumption that $|\omega-\omega_c|\gg\nu$ or ω_k. Therefore, further simplification can be made to obtain the leading terms in the case of ω, $\omega_c\gg\omega_k$. Thus, we can ignore ω_k in the dielectric functions in Eqs. (33), (34). Also, we make the substitutions that

$$|v_k|^2\approx(4\pi e^2/k)^2(n_0/M),$$

and $\omega_k=ck$, where c, the sound speed, is given by $c=(m/3M)^{1/2}v_F$ and M is the ion mass. Then Eqs. (33), (34) become

$$\tilde{I}_1(\omega)=\sum_\mathbf{k}\frac{2\pi k_{||}^2}{Mck}\coth\left(\frac{\beta ck}{2}\right)\left\{\frac{1}{D(\mathbf{k},0)}-\frac{1}{D(\mathbf{k},\omega)}\right\}, \quad (35)$$

$$\tilde{I}_2(\omega)=\sum_\mathbf{k}\frac{\pi k_\perp^2}{Mck}\coth\left(\frac{\beta ck}{2}\right)\left\{\frac{1}{D(\mathbf{k},0)}-\frac{1}{D(\mathbf{k},\omega)}\right\}. \quad (36)$$

We can now obtain the frequency-dependent relaxation time via the definitions

$$\sigma_{||}=\frac{ie^2 n_0}{m_{||}^*}\frac{1}{\omega+i/\tau_{||}},$$

$$\sigma_\pm=\frac{ie^2 n_0}{m_\pm^*}\left(1\pm\frac{\omega_c}{\omega}\right)^{-1}\frac{1}{\omega+i/\tau_\pm}, \quad (37)$$

where

$$m_{||}^*=m\left[1+\frac{1}{m\omega^2 n_0}\operatorname{Re}\tilde{I}_1(\omega)\right],$$

$$m_\pm^*=m\left[1+\frac{1}{m\omega^2 n_0}\left(1\pm\frac{\omega_c}{\omega}\right)^{-1}\operatorname{Re}\tilde{I}_2(\omega)\right], \quad (38)$$

and

$$\tau_{||}^{-1}=\frac{1}{m\omega n_0}\operatorname{Im}\tilde{I}_1(\omega),$$

$$\tau_\pm^{-1}=\frac{1}{m\omega n_0}\left(1\pm\frac{\omega_c}{\omega}\right)^{-1}\operatorname{Im}\tilde{I}_2(\omega). \quad (39)$$

Let us examine some of the features of these results which we presume are valid in general.[1] Irrespective of the magnetic field, for normal metals where $kT\ll E_F$, $D(\mathbf{k},\omega)$ is essentially independent of temperature and the temperature dependence of the relaxation time comes from the $\coth(\beta ck/2)$ factor in Eqs. (35), (36). For temperatures much below the Debye temperature, $\coth(\beta ck/2)$ approaches unity. The lifetimes are finite and proportional to $M^{1/2}$. They do not approach zero as in the low-frequency case due to the possibility of the electrons to absorb photons and emit phonons. For temperatures above the Debye temperature, $\coth(\beta ck/2)$ approaches $2/\beta ck$. Therefore, the relaxation times are proportional to β and independent of the ion mass. Aside from the factors $(1\pm\omega_c/\omega)^{-1}$, the conductivity

tensor depends on the magnetic field through $D(\mathbf{k},0)$ and $D(\mathbf{k},\omega)$. These dielectric functions are expressed by an infinite sum as can be seen from Eqs. (21), (22) and have to be evaluated numerically.

IV. CONCLUSION

In arriving at our final results, Eqs. (37), (38), and (39), we have assumed [Eq. (29)] that the phonons are in thermal equilibrium. This implies that the electron density fluctuates so fast at a frequency ω much higher than ω_k and therefore the phonons cannot follow the fluctuations.

Our result for the longitudinal conductivity is therefore valid when $\omega \gg \omega_k$. For the transverse conductivity our results are valid for $|\omega - \omega_c| \gg \omega_k$. which includes the case of low applied frequency ω but $\omega_c \gg \omega_k$.

Elastic Constants of Barium Fluoride Between 4.2 and 300°K

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The elastic constants of single-crystal barium fluoride have been determined in the 4.2–300°K temperature range, by an ultrasonic pulse-echo technique. The values of the elastic constants, extrapolated to 0°K, in units of 10^{11} dyn/cm², are: $c_{11}=9.810$, $c_{12}=4.481$, and $c_{44}=2.544$. The constants c_{11}, and c_{12} change by about 10%, while c_{44} hardly changes over the above temperature range. Barium fluoride is found to be mechanically isotropic, the value of the anisotropy factor $2c_{44}(c_{11}-c_{12})^{-1}$ being close to unity between 4.2 and 300°K. A value of 282°K for the Debye temperature at 0°K is computed from the elastic constants. The values of the elastic constants are deduced theoretically from a rigid-ion model, and the results compared with the experimental data.

I. INTRODUCTION

THE alkaline-earth fluorides have been extensively investigated over the last years, especially their optical properties[1] and the behavior of paramagnetic-ion impurities in their host lattices.[2] As they form ionic crystals, and thus their lattice forces might be derived theoretically from a simple rigid-ion model,[3] it is of interest to investigate their mechanical properties. The elastic constants of CaF_2 from 4.2 to 300°K have been determined recently,[4] and the present report describes measurements of the adiabatic elastic-stiffness constants of BaF_2 over the temperature range 4.2–300°K. Measurements of elastic constants, and their temperature dependence provide information about the mechanical and thermodynamic properties of the lattice. They can also be compared with the theoretically calculated values, and thus the validity of the assumptions underlying the theoretical model can be examined.

II. EXPERIMENTAL TECHNIQUE

Barium fluoride is a cubic crystal, and thus it has three independent elastic constants: c_{11}, c_{12}, and c_{44}. The latter were determined by measuring the sound velocity in different crystalline directions, by a pulse-echo technique, utilizing unrectified pulses at 15 Mc/sec.[5] Such a technique avoids the necessity for the "time-of-flight correction," and also increases the accuracy. A single crystal of BaF_2 was oriented by means of x-ray Laue back reflection, and two pairs of parallel surfaces were ground on it. One set of surfaces corresponded to a (111) crystalline plane, while the other set was a (110). X- and Y-cut quartz transducers of 5-Mc/sec fundamental frequency, operating at their third harmonic, were used in generating longitudinal and shear sound waves, respectively. For the room-temperature measurements, phenyl salicylate (salol) was used in bonding the transducer to the sample, glycerine, and "Nonaq" stopcock grease were used from below room temperature, to about 120°K. From there, down to 4.2°K, Dow-Corning No. 200 silicone fluid, 1000-cstoke viscosity, was used as a bonding agent. Where measurements with different bonding materials were made in overlapping temperature ranges, the agreement of the results was within the limit of error. After bonding the transducer, the crystal was placed inside a cryostat, under a dry helium atmosphere, where its temperature could be varied between 4.2 and 300°K.

The elastic constants were determined by measuring the sound velocity of five different propagation modes. These were a longitudinal and shear wave in the [111] direction, and a longitudinal and two shear

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